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An asymptotic analysis of the three-dimensional displacements and stresses in a spherical shell under inward radially opposed concentrated surface loads

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Abstract

This paper complements and extends a recent asymptotic treatment of the title problem by Gregory et al. (SIAM J. Appl. Math. 59 (1999) 1080) who considered those solutions of the three-dimensional elasticity equations for an isotropic spherical shell of constant thickness $2H$ that can be identified as membrane-like or shell-like. No attempt was made to analyze the solutions of the governing equations in neighborhoods of radius $O(H)$ of the concentrated surface loads, i.e., three-dimensional slab-like solutions. Herein, formal asymptotic solutions are constructed for the shell-like and slab-like solutions. (The membrane-like solutions of Gregory et al. are exact, simple, and explicit and require no asymptotic treatment.) The analysis in the present paper reveals clearly how the three types of solutions blend into one another and allows one to assess the errors in classical (Kirchhoff–Love) shell theory. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Exact three-dimensional solutions of shell-like bodies, though rare, are a valuable source of counter-examples or conjectures concerning claims of the accuracy of two-dimensional shell theories. Such claims become particularly questionable and difficult to assess near geometric or load discontinuities (edges or points of application of concentrated loads, for example). The relative simplicity of the governing three-dimensional equations of linear elasticity for a symmetrically loaded, elastically isotropic spherical shell make it feasible to seek exact solutions. For example, Vilenskaia and Vorovich (1966) obtain closed-form solutions for the eigensolutions (eigenvalues and eigenfunctions) which satisfy the Navier equations and stress-free boundary conditions on the faces of a spherical shell. They then show that, as the relative thickness of the shell approaches zero, the eigenvalues separate into three distinct groups. If the loads along

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the edges of a spherical cap are self-equilibrating, then the first group of eigenvalues yields null-stress solutions. (The exact membrane-like solutions of Gregory et al. (1999) for a closed, point-loaded spherical shell replace this first group of null solutions.) The second and third group of eigenvalues are identified with shell-like and Saint-Venant (or slab-like) solutions. Expanding the eigensolutions in powers of a dimensionless thickness parameter $\gamma = \ln(1+\varepsilon) - \ln(1-\varepsilon)$, where $\varepsilon = H/R$, the ratio of the half thickness to the mid-surface radius, Vilenskaia and Vorovich compute (in exquisite, pre-mathematica detail) the mid-surface displacements in a closed spherical shell subject to self-equilibrating surface tractions and compare their results with those of Vlasov's shell theory. They also present series expansions for an open spherical cap subject to self-equilibrating edge tractions. For other related Soviet work, see Vorovich (1975) and the references therein.

Apparently unaware of the work of Vilenskaia and Vorovich (1966), Cheng and Angsirikul (1977) also consider exact three-dimensional solutions for a spherical cap under axisymmetric, self-equilibrated edge tractions. As Cheng and Angsirikul observe, the associated equation for the eigenvalues "... is indeed very difficult to solve analytically or even numerically ...". They therefore work with various truncations to obtain approximate representations of the stresses prescribed along the edge of the cap. Note that neither Vilenskaia and Vorovich nor Cheng and Angsirikul consider concentrated loads.

Finally, we mention the work of Gregory et al. (1999) who obtain simple, exact, closed-form membrane-like solutions and exact eigenfunction expansions for the shell-like solutions for a closed spherical under equal and opposite, radially outward surface point loads. Moreover, using the Betti Reciprocity Principle, they obtain an explicit formula for the expansion coefficients (up to exponentially small terms). The shell-like solutions are then expanded in powers of ε to obtain corrections to classical (Kirchhoff–Love) thin shell theory. However, they make no attempt to find the elasticity solutions in the immediate vicinity of the point loads "Because ... the stress singularities at the load points [make] the Saint Venant component ... extremely complex and difficult to calculate."

The present paper takes a different approach in the spirit of the three-dimensional perturbation analyses of Johnson and Reissner (1958) for an end-loaded, semi-infinite circular cylindrical shell and of Cole (1968) for a clamped, pressurized spherical cap. (Aside from these two relevant papers, there is an extensive literature – that we make no attempt to cite – going back at least to Goodier (1938), on deriving beam, plate, and shell theories from the equations of three-dimensional elasticity via asymptotic methods.) With the exception of the simple exact membrane-like solution found by Gregory et al. (1999), we look for asymptotic expansions of the governing *equations* rather than of the exact *solutions*. Thus, we exploit the well-known observation that, if a differential equation contains a small parameter, it is almost always simpler to obtain an asymptotic expansion of the solution by working directly with the differential equation itself than with its solution. (Indeed, for many differential equations, there may be no convenient representation for the solution from which to extract an asymptotic expansion.)

In particular, we modify and extend the analysis of Gregory et al. (1999) by analyzing asymptotically the behavior of the shell (a) in a shallow region near the concentrated loads but outside a cylindrical neighborhood of radius O(H) and (b) in the immediate vicinity of the concentrated loads. We show, as is expected, that, to a first approximation, linear shallow shell theory holds in (a) and that the three-dimensional elasticity equations for a slab hold in (b). This allows us to use the analysis of Simmonds (1990) for the latter.

2. The governing equations

Let P denote the magnitude of each of the two opposed *inward* radial surface point loads and, in a system of spherical coordinates, $(R\rho, \phi, \theta)$, let $(P/GH)(u, w)$ denote the meridional and outward radial components of displacement and $(P/HR\rho)(\sigma_\rho, \sigma_\theta, \sigma_\phi, \tau)$ denote the non-vanishing physical components of the stress tensor, [1] where G is the shear modulus. The governing dimensionless equations, taken from Sokolnikoff (1956, p. 184), but in our notation, comprise the two equilibrium equations,

$$(\rho\sigma_\rho)_{,\rho} + \tau_{,\phi} - \sigma_\theta - \sigma_\phi + \tau\cot\phi = 0, \tag{2.1}$$

$$(\rho\tau)_{,\rho} + \sigma_{\phi,\phi} + (\sigma_\phi - \sigma_\theta)\cot\phi + \tau = 0, \tag{2.2}$$

and the four stress–displacement relations

$$2(1+v)\rho w_{,\rho} = \sigma_\rho - v(\sigma_\theta + \sigma_\phi), \tag{2.3}$$

$$(1-v)\sigma_\theta = 2[(1+v)w + u\cot\phi + vu_{,\phi}] + v\sigma_\rho, \tag{2.4}$$

$$(1-v)\sigma_\phi = 2[(1+v)w + u_{,\phi} + vu\cot\phi] + v\sigma_\rho, \tag{2.5}$$

$$w_{,\phi} + \rho u_{,\rho} - u = \tau. \tag{2.6}$$

In these equations, a comma denotes differentiation with respect to the subscript that follows and v is Poisson's ratio.

The boundary conditions are that the inner and outer surfaces of the shell be stress free, except for the concentrated loads at $\rho = 1 + \varepsilon$, $\phi = 0$, π. These loads may be accounted for by delta functions or equivalently by equating the vertical force on an arbitrary spherical cap of angular width ϕ to the downward load P. In spherical coordinates these conditions, in dimensionless form, read

$$\sigma_\rho(1\pm\varepsilon,\phi) = \tau(1\pm\varepsilon,\phi) = 0, \quad 0 < \phi < \pi \tag{2.7}$$

and

$$2\pi\sin\phi\int_{1-\varepsilon}^{1+\varepsilon}[\tau(\rho,\phi)\cos\phi - \sigma_\phi(\rho,\phi)\sin\phi]\mathrm{d}\rho = \varepsilon, \quad 0 < \phi < \pi. \tag{2.8}$$

2.1. The exact equilibrium equations of classical shell theory

Because classical shell theory involves only stress resultants and couples, which fall out naturally when we perform certain (weighted) integrations through the thickness of the local three-dimensional equilibrium equations, a meaningful discussion of pointwise errors reduces to examining the first-order corrections to these quantities. Thus, integrating Eqs. (2.1) and (2.2) with respect to ρ from $1-\varepsilon$ to $1+\varepsilon$ and noting the traction-free face conditions (2.7), we obtain

$$\varepsilon^{1/2}(Q^\bullet + Q\cot\phi) = n_\theta + n_\phi \tag{2.9}$$

and

$$n_\phi^\bullet + (n_\phi - n_\theta)\cot\phi + \varepsilon^{1/2}Q = 0, \tag{2.10}$$

where a dot ($^\bullet$) denotes differentiation with respect to ϕ and

$$\{n_\theta, n_\phi, \varepsilon^{1/2}Q\} \equiv \int_{1-\varepsilon}^{1+\varepsilon}\{\sigma_\theta, \sigma_\phi, \tau\}\,\mathrm{d}\rho. \tag{2.11}$$

The physical stress resultants are $(P/H)\{n_\theta, n_\phi, \varepsilon^{1/2}Q\}$.

[1] Because of the simplifying factor ρ that we have introduced in these definitions, we call σ_ρ, etc., *pseudo*-stress components.

The single dimensionless moment equilibrium equation of classical shell theory is obtained by multiplying Eq. (2.2) by $\rho - 1$ and again integrating with respect to ρ from $1-\varepsilon$ to $1+\varepsilon$. Integrating the term $(\rho - 1)(\rho\tau)_{,\rho}$ by parts and noting the second face condition in Eq. (2.7), we have

$$\varepsilon^{1/2}[M_\phi^\bullet + (M_\phi - M_\theta)\cot\phi] = Q, \tag{2.12}$$

where

$$\varepsilon\{M_\theta, M_\phi\} \equiv \int_{1-\varepsilon}^{1+\varepsilon} (\rho - 1)\{\sigma_\theta, \sigma_\phi\}\mathrm{d}\rho. \tag{2.13}$$

The physical stress couples are $P\{M_\theta, M_\phi\}$. The scalings introduced in the definitions (2.11) and (2.13) guarantee that in shallow regions of the mid-surface, near to but excluding a neighborhood of the poles, Q, M_θ, and M_ϕ are all O(1).

If we combine Eqs. (2.9) and (2.10) to form an equation of local vertical equilibrium, we obtain, after an integration,

$$2\pi\sin\phi(\varepsilon^{1/2}Q\cos\phi - n_\phi\sin\phi) = \varepsilon. \tag{2.14}$$

This equation is simply Eq. (2.8) with the definitions (2.11) inserted.

3. The exact membrane-like solutions

From Eqs. (B.1)–(B.6) of Gregory et al. (1999), we have

$$8\pi u^M = \cot\phi + \sin\phi\ln(\cot\phi/2), \qquad 8\pi w^M = 1 + \cos\phi\ln(\tan\phi/2), \tag{3.1}$$

and

$$\sigma_\rho^M = \tau^M = 0, \qquad 4\pi\sigma_\theta^M = -4\pi\sigma_\phi^M = \csc^2\phi. \tag{3.2}$$

Note that the membrane-like stresses satisfy the stress-free face boundary conditions exactly and that the membrane displacements and stresses have the opposite sign of those of Gregory and Wan because we have taken the concentrated loads to be positive *inwards*. Further, note by Eqs. (2.11) and (2.13) that

$$\{n_\theta^M, n_\phi^M, Q^M, M_\theta^M, M_\phi^M\} = (\varepsilon/2\pi)\csc^2\phi\{1, -1, 0, 0, 0\}. \tag{3.3}$$

3.1. Physical meaning of the three-dimensional membrane-like solution

Although the dimensionless membrane-like displacements and pseudo-stresses satisfy the field equations and traction-free face conditions of three-dimensional elasticity, their singularities are too strong in neighborhoods of the poles. For example, at the north pole ($\phi = 0$), the membrane-like pseudo-stresses are *not* equilibrating a concentrated surface load but rather an *inward radial body force* $(P/HR^2\rho^2)\delta(\phi)/4\pi\phi$, where δ is the delta function(al). The function of the shell-like portion of the solution, that we consider next, is to provide a transition whereby the membrane-like tangential forces near the poles are converted into a statically equivalent vertical force, thus producing milder singularities as $\phi \to 0, \pi$.

4. The shell-like solution

Let

$$n_\theta = n_\theta^M + N_\theta \quad \text{and} \quad n_\phi = n_\phi^M + N_\phi. \tag{4.1}$$

Then Eqs. (2.9), (2.10), and (2.14) read

$$\varepsilon^{1/2}(Q^\bullet + Q\cot\phi) = N_\theta + N_\phi, \tag{4.2}$$

$$N_\phi^\bullet + (N_\phi - N_\theta)\cot\phi + \varepsilon^{1/2}Q = 0 \tag{4.3}$$

and

$$\varepsilon^{1/2}Q = N_\phi \tan\phi. \tag{4.4}$$

Substituting Eq. (4.4) into Eq. (4.3) and multiplying the resulting expression by $\tan\phi$, we obtain

$$(N_\phi \sec\phi)^\bullet \sin\phi = N_\theta - N_\phi \text{ or } N_\theta = (N_\phi \tan\phi)^\bullet. \tag{4.5}$$

We may reduce these basic shell equations (which, as yet, involve no approximations) still further as follows.

Let

$$\{\varepsilon^{1/2}\underline{u}, \underline{w}\} \equiv (1/2)\int_{1-\varepsilon}^{1+\varepsilon} \{u(\rho,\phi;\varepsilon) - u^M(\phi), w(\rho,\phi;\varepsilon) - w^M(\phi)\}\mathrm{d}\rho \tag{4.6}$$

and

$$\{\varepsilon^{3/2}\underline{\underline{u}}, \varepsilon^2\underline{\underline{w}}\} \equiv (3/2)\int_{1-\varepsilon}^{1+\varepsilon} (\rho - 1)\{u(\rho,\phi;\varepsilon), w(\rho,\phi;\varepsilon)\}\mathrm{d}\rho. \tag{4.7}$$

The three-dimensional displacement–stress relations (2.4) and (2.5) then imply the shell constitutive relations,

$$(1-v)N_\theta = 4[(1+v)\underline{w} + \varepsilon^{1/2}(\underline{u}\cot\phi + v\underline{u}^\bullet)] + v\varepsilon N_\rho, \tag{4.8}$$

$$(1-v)N_\phi = 4[(1+v)\underline{w} + \varepsilon^{1/2}(\underline{u}^\bullet + v\underline{u}\cot\phi)] + v\varepsilon N_\rho, \tag{4.9}$$

$$(1-v)M_\theta = (4/3)[(1+v)\varepsilon\underline{\underline{w}} + \varepsilon^{1/2}(\underline{\underline{u}}\cot\phi + v\underline{\underline{u}}^\bullet)] + v\varepsilon M_\rho, \tag{4.10}$$

$$(1-v)M_\phi = (4/3)[(1+v)\varepsilon\underline{\underline{w}} + \varepsilon^{1/2}(\underline{\underline{u}}^\bullet + v\underline{\underline{u}}\cot\phi)] + v\varepsilon M_\rho, \tag{4.11}$$

where

$$\varepsilon N_\rho \equiv \int_{1-\varepsilon}^{1+\varepsilon} \sigma_\rho(\rho,\phi;\varepsilon)\mathrm{d}\rho \quad \text{and} \quad \varepsilon^2 M_\rho \equiv \int_{1-\varepsilon}^{1+\varepsilon} (\rho - 1)\sigma_\rho(\rho,\phi;\varepsilon)\mathrm{d}\rho. \tag{4.12}$$

If we substitute Eqs. (4.8) and (4.9) into Eq. (4.5), we obtain a differential equation involving $\underline{u}$, and N_ϕ that may be written as a total derivative. Integrating and discarding an inessential constant, we find that

$$4\varepsilon^{1/2}\underline{u} = -N_\phi \tan\phi, \tag{4.13}$$

and thus, by Eqs. (4.4) and (4.5), we have the remarkably simple expression for the dimensionless hoop and transverse shear stress resultants,

$$N_\theta = -4\varepsilon^{1/2}\underline{u}^\bullet \quad \text{and} \quad Q = -4\underline{u}. \tag{4.14}$$

Substituting the second term of Eq. (4.14) along with Eqs. (4.10) and (4.11) into Eq. (2.12), we obtain the following explicit formula for $\underline{u}$ in terms of $\underline{\underline{u}}$, $\underline{\underline{w}}$, and M_ρ,

$$3(1-\nu)\underline{u} = -\varepsilon[(1+\nu)\varepsilon^{1/2}\underline{\underline{w}}^{\bullet} + \underline{\underline{u}}^{\bullet\bullet} + \underline{\underline{u}}^{\bullet}\cot\phi - \underline{\underline{u}}\cot^2\phi - \nu\underline{\underline{u}} + (3/4)\nu\varepsilon^{1/2}M_\rho^{\bullet}]. \tag{4.15}$$

An additional relation among $\underline{u}$, $\underline{w}$, and N_ρ follows on substituting Eqs. (4.8), (4.9), and (4.14) into Eq. (4.2):

$$\varepsilon^{1/2}(\underline{u}\sin\phi)^{\bullet} = -[(1+\nu)\underline{w} + (\nu/4)\varepsilon N_\rho]\sin\phi. \tag{4.16}$$

This relation, in turn, implies that

$$N_\theta + N_\phi = 4(1+\nu)\underline{w} + \nu\varepsilon N_\rho. \tag{4.17}$$

To proceed further, we must now perform asymptotic integration in the thickness direction. To this end, we introduce the scaled independent variables

$$\phi = \varepsilon^{1/2}\alpha, \qquad \rho = 1 + \varepsilon\zeta, \tag{4.18}$$

and set

$$u = u^M(\rho,\phi) + \varepsilon^{-1/2}U(\alpha,\zeta;\varepsilon), \qquad w = w^M(\rho,\phi) + \varepsilon^{-1}W(\alpha,\zeta;\varepsilon), \tag{4.19}$$

$$\sigma_\rho = S_\rho(\alpha,\zeta;\varepsilon), \qquad \tau = \varepsilon^{-1/2}T(\alpha,\zeta;\varepsilon), \tag{4.20}$$

$$\sigma_\theta = \sigma_\theta^M(\rho,\phi) + \varepsilon^{-1}S_\theta(\alpha,\zeta;\varepsilon), \qquad \sigma_\phi = \sigma_\phi^M(\rho,\phi) + \varepsilon^{-1}S_\phi(\alpha,\zeta;\varepsilon). \tag{4.21}$$

Rearranging Eqs. (2.1)–(2.6) in the form and order in which we shall solve them, we have

$$\begin{aligned} 2(1+\nu)W_{,\zeta} &= -\varepsilon[\nu(S_\theta + S_\phi) + 2(1+\nu)\zeta W_{,\zeta}] + \varepsilon^2 S_\rho \\ &= -\nu\varepsilon(S_\theta + S_\phi) + \varepsilon^2[S_\rho + \nu\zeta(S_\theta + S_\phi) + 2(1+\nu)\zeta^2 W_{,\zeta}] - \varepsilon^3\zeta S_\rho, \end{aligned} \tag{4.22}$$

$$U_{,\zeta} = -W_{,\alpha} + \varepsilon(U - \zeta U_{,\zeta} + T) = -W_{,\alpha} + \varepsilon(\zeta W_{,\alpha} + U + T) - \varepsilon^2\zeta(U - \zeta U_{,\zeta} + T), \tag{4.23}$$

$$(1-\nu)S_\theta = 2[(1+\nu)W + \alpha^{-1}U\,\mathrm{cotc}(\varepsilon^{1/2}\alpha) + \nu U_{,\alpha}] + \nu\varepsilon S_\rho, \tag{4.24}$$

$$(1-\nu)S_\phi = 2[(1+\nu)W + U_{,\alpha} + \nu\alpha^{-1}U\,\mathrm{cotc}(\varepsilon^{1/2}\alpha)] + \nu\varepsilon S_\rho, \tag{4.25}$$

$$T_{,\zeta} = -S_{\phi,\alpha} + \alpha^{-1}(S_\theta - S_\phi)\mathrm{cotc}(\varepsilon^{1/2}\alpha) - \varepsilon(\zeta T_{,\zeta} + 2T), \tag{4.26}$$

$$S_{\rho,\zeta} = S_\theta + S_\phi - [T_{,\alpha} + \alpha^{-1}T\,\mathrm{cotc}(\varepsilon^{1/2}\alpha)] - \varepsilon(\zeta S_\rho)_{,\zeta}, \tag{4.27}$$

where

$$\mathrm{cotc}\,\phi = \frac{\cos\phi}{\mathrm{sinc}\,\phi} \quad \text{and} \quad \mathrm{sinc}\,\phi \equiv \begin{cases} \phi^{-1}\sin\phi, & \phi \neq 0 \\ 1, & \phi = 0 \end{cases}. \tag{4.28}$$

The traction-free face conditions (2.7) read

$$S_\rho(\alpha,\pm 1;\varepsilon) = T(\alpha,\pm 1;\varepsilon) = 0, \quad 0 < \alpha < \varepsilon^{-1/2}\pi. \tag{4.29}$$

The self-equilibrating outward radial body force $(P/HR^2\rho^2)F(\zeta;\varepsilon)\delta(\alpha)/2\pi\alpha$ that produces the shell-like solutions discussed in this section, can be expressed in terms of limiting values of the dimensionless shell-like stresses if we consider the equilibrium of the shell segment $0 \leqslant \alpha < \beta$, $-1 \leqslant \zeta \leqslant \xi$. Assuming $\alpha[S_\rho(\alpha,\zeta;\varepsilon) - \varepsilon\alpha T(\alpha,\zeta;\varepsilon)]$ bounded, we have, because the inner surface of the shell is traction free,

$$\int_{-1}^{\xi}\{2\pi\lim_{\alpha\to 0}\alpha[T(\alpha,\zeta;\varepsilon)-\alpha S_{\phi}(\alpha,\zeta;\varepsilon)]+F(\zeta;\varepsilon)\}\mathrm{d}\zeta=0,\quad -1\leqslant\xi\leqslant 1. \tag{4.30}$$

Thus, if the integrand is continuous,

$$F(\zeta;\varepsilon)=2\pi\lim_{\alpha\to 0}\alpha[\alpha S_{\phi}(\alpha,\zeta;\varepsilon)-T(\alpha,\zeta;\varepsilon)]. \tag{4.31}$$

We now assume that our dimensionless unknowns $U,\dots,T$ have asymptotic expansions of the form

$$f=\overset{0}{f}(\alpha,\zeta)+\varepsilon\overset{1}{f}(\alpha,\zeta)+\mathrm{O}(\varepsilon^2). \tag{4.32}$$

5. The lowest-order equations

From Eq. (4.22), $\overset{0}{W}_{,\zeta}=0$. Hence,

$$\overset{0}{W}=\overset{0}{w}(\alpha), \tag{5.1}$$

where $\overset{0}{w}$ is unknown. Next, from Eq. (4.23), $\overset{0}{U}_{,\zeta}=-\overset{0}{w}'(\alpha)$. Hence,

$$\overset{0}{U}=\overset{0}{u}(\alpha)-\zeta\overset{0}{w}'(\alpha), \tag{5.2}$$

where $\overset{0}{u}$ is unknown and a prime ($'$) denotes differentiation with respect to α. Note that our formal, lowest order, asymptotic expansions (5.1) and (5.2) deliver the same forms for the displacements as does the Kirchhoff hypothesis.

From Eqs. (4.6), (4.7), (4.18), (4.19), (5.1), and (5.2),

$$\overset{0}{\underline{u}}=\overset{0}{u},\qquad \overset{0}{\underline{w}}=\overset{0}{w},\qquad \overset{0}{\underline{\underline{u}}}=-\overset{0}{w}'. \tag{5.3}$$

Thus, Eqs. (4.15) and (4.16) imply that

$$3(1-\nu)\overset{0}{\underline{u}}=(\mathscr{L}\overset{0}{w})'\quad\text{and}\quad(\alpha\overset{0}{u})'=-(1+\nu)\alpha\overset{0}{w}, \tag{5.4}$$

where

$$\mathscr{L}\equiv\frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\right). \tag{5.5}$$

Eliminating $\overset{0}{\underline{u}}$ between the two equations in Eq. (5.4), we obtain the classical shallow shell equation for the axisymmetric deformation of a spherical shell (Reissner, 1946a,b),

$$\mathscr{L}^2\overset{0}{w}+k^4\overset{0}{w}=0,\quad\text{where } k^4=3(1-\nu^2). \tag{5.6}$$

This is a Bessel-type equation whose solutions may be expressed in terms of the functions ber $k\alpha$, bei $k\alpha$, ker $k\alpha$, and kei $k\alpha$. The first two of these grow exponentially as $\xi\to\imath nfty$ and must be discarded; the remaining two decay exponentially. Thus,

$$\overset{0}{\underline{w}}=\overset{0}{w}=c_1\ \mathrm{ker}\ k\alpha+c_2\,\mathrm{kei}\,k\alpha, \tag{5.7}$$

where c_1 and c_2 are unknown constants to be determined by the requirement that, to lowest order, the thickness-averaged dimensionless meridional and radial displacements, $\underline{u}^M+\varepsilon^{-1/2}\underline{u}$ and $\underline{w}^M+\varepsilon^{-1}\underline{w}$, remain finite as $\phi\to 0$.

To this end, note that

$$\mathscr{L}\ \ker k\alpha = -k^2 \operatorname{kei} k\alpha, \qquad \mathscr{L} \operatorname{kei} k\alpha = k^2 \ker k\alpha, \tag{5.8}$$

$$\ker x = -\ln(1/2x) - \gamma + (\pi/16)x^2 + \mathrm{O}(x^4 \ln x), \tag{5.9}$$

and

$$\operatorname{kei} x = -(\pi/4) - (1/4)x^2(\ln(1/2x) + 1 - \gamma) + \mathrm{O}(x^6 \ln x), \tag{5.10}$$

where $\gamma = 0.5772\ldots$ is Euler's constant. Thus, $c_1 = 0$ and, from Eqs. (5.2) and (5.3), the first term of Eq. (5.4), and Eqs. (5.6)–(5.10),

$$\overset{0}{u} = \sqrt{(1/3)(1+v)(1-v)^{-1}}\,c_2(\ker k\alpha)' = \alpha^{-1}\left[-c_2\sqrt{(1/3)(1+v)(1-v)^{-1}} + \mathrm{O}(\alpha)\right]. \tag{5.11}$$

But from the first term of Eq. (3.1) and Eq. (4.18),

$$\underline{u}^M = u^M = \varepsilon^{-1/2}[(1/8\pi)\alpha^{-1} + \mathrm{O}(\varepsilon \ln \varepsilon)]. \tag{5.12}$$

Hence,

$$c_2 = (1/8\pi)\sqrt{3(1-v)(1+v)^{-1}}, \tag{5.13}$$

so that

$$\overset{0}{w} = (1/8\pi)\sqrt{3(1-v)(1+v)^{-1}}\operatorname{kei} k\alpha = -(1/32)\sqrt{3(1-v)(1+v)^{-1}} + \mathrm{O}(\alpha^2 \ln \alpha). \tag{5.14}$$

This expression at $\alpha = 0$, when multiplied by P/GH to obtain the dimensional outward radial displacement, agrees with Eq. (5.13) of Koiter (1963).

We now compute the various dimensionless pseudo-stresses to lowest order. First, because.

$$(\mathscr{L}\overset{0}{w})'' = \mathscr{L}^2\overset{0}{w} - \alpha^{-1}(\mathscr{L}\overset{0}{w})', \tag{5.15}$$

it follows from Eqs. (4.18), (4.24), (4.25), (5.1)–(5.4), and (5.6) that

$$(1-v)\overset{0}{S}_\theta = 2\left[(1-v^2)\overset{0}{w} + (1/3)\alpha^{-1}(\mathscr{L}\overset{0}{w})' - \zeta(\alpha^{-1}\overset{0}{w}{}' + v\overset{0}{w}{}'')\right] \tag{5.16}$$

and

$$(1-v)\overset{0}{S}_\phi = -2\left[(1/3)\alpha^{-1}(\mathscr{L}\overset{0}{w})' + \zeta(\overset{0}{w}{}'' + v\alpha^{-1}\overset{0}{w}{}')\right]. \tag{5.17}$$

Substituting these expressions into the lowest-order form of Eq. (4.26), we find that

$$(1/2)(1-v)\overset{0}{T}_{,\zeta} = \zeta\left(\mathscr{L}\overset{0}{w}\right)'. \tag{5.18}$$

Integrating Eq. (5.18) with respect to ζ and applying the traction-free face condition in the second term of Eq. (4.29), we obtain

$$(1-v)\overset{0}{T} = (\zeta^2 - 1)\left(\mathscr{L}\overset{0}{w}\right)'. \tag{5.19}$$

To complete our lowest order approximations, note from Eqs. (4.18) and (4.27) that

$$\overset{0}{S}_{\rho,\zeta} = \overset{0}{S}_\theta + \overset{0}{S}_\phi - \alpha^{-1}(\alpha\overset{0}{T})_{,\alpha}. \tag{5.20}$$

But, from Eqs. (5.16) and (5.17),

$$\overset{0}{S}_\theta + \overset{0}{S}_\phi = 2(1+v)[\overset{0}{w} - (1-v)^{-1}\zeta\mathscr{L}\overset{0}{w}] \tag{5.21}$$

and, by Eqs. (5.5), (5.6), and (5.19),

$$\alpha^{-1}(\alpha\overset{0}{T})_{,\alpha} = (1-v)^{-1}(\zeta^2-1)\mathscr{L}^2\overset{0}{w} = 3(1+v)(1-\zeta^2)\overset{0}{w}. \tag{5.22}$$

Hence,

$$\overset{0}{S}_{\rho,\zeta} = -(1+v)[\overset{0}{w} + 2(1-v)^{-1}\zeta\mathscr{L}\overset{0}{w} - 3\zeta^2\overset{0}{w}]. \tag{5.23}$$

Integrating and choosing the unknown function of integration so that the face condition in the first term of Eq. (4.29) is satisfied (to lowest order), we find that

$$\overset{0}{S}_\rho = (1+v)(\zeta^2-1)[\zeta\overset{0}{w} - (1-v)^{-1}\mathscr{L}\overset{0}{w}]. \tag{5.24}$$

Finally, from Eqs. (4.8)–(4.11), (4.18), and (5.4), we have

$$(1-v)\overset{0}{N}_\theta = -(4/3)(\mathscr{L}\overset{0}{w})'' = 4(1-v^2)\overset{0}{w} + (4/3)\alpha^{-1}(\mathscr{L}\overset{0}{w})', \tag{5.25}$$

$$(1-v)\overset{0}{N}_\phi = -(4/3)\alpha^{-1}(\mathscr{L}\overset{0}{w})', \tag{5.26}$$

$$(1-v)\overset{0}{M}_\theta = -(4/3)(\alpha^{-1}\overset{0}{w}' + v\overset{0}{w}''), \tag{5.27}$$

$$(1-v)\overset{0}{M}_\phi = -(4/3)(\overset{0}{w}'' + v\alpha^{-1}\overset{0}{w}'). \tag{5.28}$$

These are the classical formulas for shallow spherical shells (Reissner, 1946a,b). Note from Eqs. (4.17), (5.27), and (5.28) the following simple expressions for the lowest-order approximations to the traces of the dimensionless stress resultants and couples:

$$\overset{0}{N}_\theta + \overset{0}{N}_\phi = 4(1+v)\overset{0}{w} \sim -(1/8)\sqrt{3(1-v^2)} \quad \text{as } \alpha \to 0 \tag{5.29}$$

and

$$(1-v)(\overset{0}{M}_\theta + \overset{0}{M}_\phi) = -(4/3)(1+v)\mathscr{L}\overset{0}{w} \sim (1/2)(1-v^2)\ln\alpha \quad \text{as } \alpha \to 0. \tag{5.30}$$

From Eqs. (4.31), (5.8)–(5.10), (5.14), (5.17), and (5.19), we find that the thickness distribution of the self-equilibrating, shell-like line load is, to lowest order,

$$\overset{0}{F}(\zeta) = (3/4)(\zeta^2-1) + (1/2), \tag{5.31}$$

where the last term in Eq. (5.31) represents that part of the self-equilibrating body force that cancels the line load producing the membrane-like solutions of Section 3.

6. The first-order approximation (correction to classical shell theory)

The only additional quantities needed to compute the dimensionless stress resultants and couples to second-order are $\overset{1}{U}$ and $\overset{1}{W}$. From Eqs. (4.22) and (5.21),

$$\overset{1}{W}_{,\zeta} = -\nu[\overset{0}{w} - (1-\nu)^{-1}\zeta\mathscr{L}\overset{0}{w}]. \tag{6.1}$$

Hence,

$$\overset{1}{W} = \overset{1}{w}(\alpha) - \nu[\zeta\overset{0}{w}(\alpha) - (1/2)(1-\nu)^{-1}\zeta^2\mathscr{L}\overset{0}{w}(\alpha)], \tag{6.2}$$

where $\overset{1}{w}$ is unknown. Next, from Eqs. (4.23), (5.1), (5.2), (5.4), (5.19), and (6.2),

$$\overset{1}{U}_{,\zeta} = -\overset{1}{W}_{,\alpha} + \zeta\overset{0}{W}_{,\alpha} + \overset{0}{U} + \overset{0}{T} = [-\overset{1}{w} - (2/3)(1-\nu)^{-1}\mathscr{L}\overset{0}{w} + \nu\zeta\overset{0}{w} + (1/2)(1-\nu)^{-1}(2-\nu)\zeta^2\mathscr{L}\overset{0}{w}]'$$
$$\equiv (\overset{1}{A} + 2\zeta\overset{1}{B} + 3\zeta^2\overset{1}{C})'. \tag{6.3}$$

Integrating, we have

$$\overset{1}{U} = \overset{1}{u}(\alpha) + [\zeta\overset{1}{A}(\alpha) + \zeta^2\overset{1}{B}(\alpha) + \zeta^3\overset{1}{C}(\alpha)]', \tag{6.4}$$

where $\overset{1}{u}$ is unknown.

Next, from Eqs. (4.6), (4.7), (4.12), (4.18)–(4.20), (5.2), (5.24), (6.2), and (6.4),

$$\underline{\overset{1}{u}} = \overset{1}{u} + (\nu/6)\overset{0}{w}', \qquad \underline{\overset{1}{w}} = \overset{1}{w} + (\nu/6)(1-\nu)^{-1}\mathscr{L}\overset{0}{w}, \tag{6.5}$$

$$\underline{\underline{\overset{1}{u}}} = -[\underline{\overset{1}{w}} + (1/15)(1-\nu)^{-1}(7-\nu)\mathscr{L}\overset{0}{w}]', \qquad \underline{\underline{\overset{0}{w}}} = -\nu\overset{0}{w}, \tag{6.6}$$

$$(1-\nu)\overset{0}{N}_\rho = (4/3)(1+\nu)\mathscr{L}\overset{0}{w}, \qquad \overset{0}{M}_\rho = -(4/15)(1+\nu)\overset{0}{w}. \tag{6.7}$$

We do not need $\underline{\underline{\overset{1}{w}}}$ in what follows.

We shall be content to compute the second-order corrections to the traces $N_\theta + N_\phi$ and $M_\theta + M_\phi$. Thus, from Eq. (4.17) and the first term of Eq. (6.7),

$$\overset{1}{N}_\theta + \overset{1}{N}_\phi = 4(1+\nu)[\underline{\overset{1}{w}} + (\nu/3)(1-\nu)^{-1}\mathscr{L}\overset{0}{w}] \tag{6.8}$$

and, from Eqs. (4.10), (4.11), (4.18), (5.3), (5.6), (6.6), and the second term of Eq. (6.7),

$$(1-\nu)(\overset{1}{M}_\theta + \overset{1}{M}_\phi) = (4/3)(1+\nu)[2\underline{\underline{\overset{0}{w}}} + \underline{\underline{\overset{1}{u}}}' + \alpha^{-1}\underline{\underline{\overset{1}{u}}} - (1/3)\alpha\underline{\underline{\overset{0}{u}}}] + 2\nu\overset{0}{M}_\rho$$
$$= (4/3)(1+\nu)[-\mathscr{L}\underline{\overset{1}{w}} + (1/5)(1-\nu)(7+\nu)\overset{0}{w} + (1/3)\alpha\overset{0}{w}']. \tag{6.9}$$

We lack only an equation for $\underline{\overset{1}{w}}$. Noting that

$$\alpha f'' = (\alpha f')' - f', \quad (\mathscr{L}f)' = f''' + \alpha^{-1}f'' - \alpha^{-2}f', \quad \mathscr{L}(\alpha f') = \alpha(\mathscr{L}f)' + 2\mathscr{L}f,$$
$$(\mathscr{L}f)'' = \mathscr{L}^2 f - \alpha^{-1}(\mathscr{L}f)', \quad \mathscr{L}(\alpha^2 f) = \alpha^2\mathscr{L}f + 4(\alpha f' + f), \tag{6.10}$$

and using Eq. (5.3), the first term of Eqs. (5.4) and (6.6), and the first term of Eq. (6.7), we find from Eqs. (4.15) and (4.16) that

$$3(1-\nu)\underline{\overset{1}{u}} = -(1+\nu)\underline{\underline{\overset{0}{w}}}' - (\underline{\underline{\overset{1}{u}}}'' + \alpha^{-1}\underline{\underline{\overset{1}{u}}}' - \alpha^{-2}\underline{\underline{\overset{1}{u}}}) + (1/3)\alpha\underline{\underline{\overset{0}{u}}}' + (\nu - 2/3)\underline{\underline{\overset{0}{u}}} - (3\nu/4)\overset{0}{M}_\rho'$$
$$= [\mathscr{L}\underline{\overset{1}{w}} - (1/5)(2+7\nu)(1-\nu)\overset{0}{w} - (1/3)\alpha\overset{0}{w}']' \tag{6.11}$$

and

$$\overset{1}{\underline{u}}{}' + \alpha^{-1}\overset{1}{\underline{u}} = -(1+\nu)\overset{1}{\underline{w}} + (1/3)\alpha\overset{0}{\underline{u}} - (\nu/4)\overset{0}{N}_\rho$$
$$= -(1+\nu)\overset{1}{\underline{w}} + (1/3)(1-\nu)^{-1}[(1/3)\alpha(\mathscr{L}\overset{0}{w})' - \nu(1+\nu)\mathscr{L}\overset{0}{w}]. \tag{6.12}$$

Substituting for $\overset{1}{\underline{u}}$ from Eq. (6.11) into Eq. (6.12), we have

$$\mathscr{L}^2\overset{1}{\underline{w}} + k^4\overset{1}{\underline{w}} = (4/15)(4-9\nu^2)\mathscr{L}\overset{0}{w} + (2/3)\alpha(\mathscr{L}\overset{0}{w})'. \tag{6.13}$$

The solution of this equation, which decays as $\alpha \to \infty$, has the form

$$\overset{1}{\underline{w}} = c_3 \ker k\alpha + c_4 \mathrm{kei}\, k\alpha + c_5\alpha^2\overset{0}{w} + c_6\alpha(\mathscr{L}\overset{0}{w})', \tag{6.14}$$

where $c_3, \dots, c_6$ are unknown constants.

To determine the constants associated with the particular solution, note from Eqs. (5.5) (the definition of the operator $\mathscr{L}$) and (5.6) that

$$\mathscr{L}[c_5\alpha^2\overset{0}{w} + c_6\alpha(\mathscr{L}\overset{0}{w})'] = 2(2c_5 - k^4c_6)\overset{0}{w} + (4c_5 - k^4c_6)\alpha\overset{0}{w}{}' + c_5\alpha^2\mathscr{L}\overset{0}{w}. \tag{6.15}$$

Hence,

$$(\mathscr{L}^2 + k^4)[c_5\alpha^2\overset{0}{w} + c_6\alpha(\mathscr{L}\overset{0}{w})'] = (16c_5 - 4k^4c_6)\mathscr{L}\overset{0}{w} + 8c_5\alpha(\mathscr{L}\overset{0}{w})', \tag{6.16}$$

so that, comparing this expression with Eq. (6.13), we see that

$$c_5 = 1/12 \text{ and } (1-\nu^2)c_6 = (1/45)(1+9\nu^2). \tag{6.17}$$

To determine c_3 and c_4, note by Eqs. (3.1), (5.8)–(5.10), (5.14), (6.11), and (6.14) that

$$\varepsilon^{-1/2}\overset{1}{\underline{u}}{}^M + \overset{1}{\underline{u}} \sim -(1/8\pi)\alpha\ln\alpha + \sqrt{(1/3)(1+\nu)(1-\nu)^{-1}}c_4\alpha^{-1} \quad \text{as } \alpha \to 0 \tag{6.18}$$

and

$$\overset{0}{\underline{w}}{}^M + \overset{1}{\underline{w}} \sim [(1/8\pi) - c_3]\ln\alpha \quad \text{as } \alpha \to 0. \tag{6.19}$$

Thus, if the first-order corrections to the lowest-order approximations to the thickness-averaged displacements are to be finite at the poles,

$$c_3 = 1/8\pi, \qquad c_4 = 0. \tag{6.20}$$

Finally, substituting Eq. (6.14) into Eqs. (6.8) and (6.9) and simplifying using Eqs. (5.8), (5.14), and (6.15), we arrive at the following expressions for the first-order corrections to the dimensionless traces of the stress resultants and couples:

$$\overset{1}{N}_\theta + \overset{1}{N}_\phi = (1/2\pi)[(1+\nu)^2\ker k\alpha + (1/12)(k\alpha)^2\mathrm{kei}\, k\alpha + (1/15)(1+9\nu^2)\alpha(\ker k\alpha)']$$
$$\sim -(1/2\pi)(1+\nu)^2 \ln\alpha \quad \text{as } \alpha \to 0 \tag{6.21}$$

and

$$(1-\nu)(\overset{1}{M}_\theta + \overset{1}{M}_\phi) = (4/3)[(k^2/40\pi)(11-\nu+5\nu^2)\mathrm{kei}\, k\alpha - (1/12)(1+\nu)(k\alpha)^2\ker k\alpha + (1/15)$$
$$\times (1+\nu)(1+9\nu^2)\alpha\overset{0}{w}{}'] \sim (1/120)\sqrt{3(1-\nu^2)}(11-\nu+5\nu^2) \quad \text{as } \alpha \to 0. \tag{6.22}$$

6.1. A note on singularities and shell theory

Although we have required the shell-like kinematic variables $\underline{u}^M + \varepsilon^{-1/2}\underline{u}$, $\underline{w}^M + \varepsilon^{-1}\underline{w}$, $\underline{\underline{u}}$, and $\underline{\underline{w}}$ to be finite at the poles, we are forced to accept singularities in our three-dimensional approximations to the meridional and outward radial displacements. Thus, from Eqs. (3.1), (5.8)–(5.10), (5.14), (6.2), (6.4), (6.11), (6.14), and (6.20),

$$\varepsilon^{-3/2}u = \varepsilon^{-3/2}u^M(\varepsilon^{1/2}\alpha) + \varepsilon^{-1}\overset{0}{U}(\alpha,\zeta) + \overset{1}{U}(\alpha,\zeta) + \mathrm{O}(\varepsilon) \sim \mathrm{O}(\alpha^{-1}) \quad \text{as } \alpha \to 0 \tag{6.23}$$

and

$$w = w^M(\varepsilon^{1/2}\alpha) + \varepsilon^{-1}\overset{0}{w}(\alpha) + \overset{1}{W}(\alpha,\zeta) + \mathrm{O}(\varepsilon) \sim \mathrm{O}(\ln\ \alpha) \quad \text{as } \alpha \to 0. \tag{6.24}$$

These unavoidable singularities are to be expected because the classical Boussinesq solution of isotropic linear elasticity implies that the deflection is infinite under a concentrated normal load on the surface of a spherical shell, *if viewed as a three-dimensional continuum.*

We also observe an interesting phenomenon with the stress resultants and couples: although $\overset{0}{N}_\theta + \overset{0}{N}_\phi$ is bounded, its first-order correction, $\overset{1}{N}_\theta + \overset{1}{N}_\phi$, has a logarithmic singularity as $\alpha \to 0$; the situation is reversed with the stress couples: $\overset{0}{M}_\theta + \overset{0}{M}_\phi$ has a logarithmic singularity as $\alpha \to 0$ whereas its first-order correction is bounded. However, one must not put too much faith in these singularities, because all bets are off as we enter a neighborhood where $\alpha = \mathrm{O}(\varepsilon^{1/2})$ near the concentrated body forces.

The virtue of shell theory is that, by working with unknowns that are through-the-thickness integrals, the singularities of three-dimensional elasticity are ameliorated, although, of course, not eliminated altogether. Physically, a linearly elastic solid cannot sustain a point load: in practice the load will be distributed over a small area and inelastic behavior may occur. By not attempting to provide a picture of thickness distributions, which cannot be accurate anyway except in a mathematical sense (e.g., as Green's functions), shell theory, in a way, gives a more physically satisfying, if limited, picture of the behavior of an elastic structure under concentrated loading.

7. Slab-like solutions

Recall that we may regard the solution of our three-dimensional linear elasticity problem as the sum of the solutions of three *exact* subproblems: (1) the membrane-like solution produced by a uniform line load along the polar axis of the shell; (2) a shell-like solution produced by a body force comprising the negative of the membrane-like line load plus an equilibrating line load having a thickness distribution along the polar axis given, to lowest order, by Eq. (5.31); and (3) a slab-like solution produced by the two concentrated polar surface loads, each of which is equilibrated by the unknown, non-uniform portion of the shell-like line load. We now turn to the determination of this latter solution.

As we have noted, very near the poles, the effect of curvature on the behavior of the spherical shell is secondary and, to a first approximation, slab-like behavior is expected. To tie-in with Simmonds' (1990) analysis of the behavior of an infinite plate under vertical, self-equilibrating surface and line loads, we introduce circular cylindrical coordinates (Rr, Rz, θ) and express the horizontal and vertical components of displacements, $(P/GH)(u_r, u_z)$, and the non-vanishing physical components of the stress tensor, $(P/H^2)(\Sigma_\theta, \Sigma_r, \Sigma_z, \Sigma)$, [2] in terms of a stress function (Love, 1944, p. 276) as

[2] These stresses are non-dimensionalized in a different way from those in Section 2.

$$2\varepsilon u_r = -\chi_{,rz}, \qquad 2\varepsilon u_z = 2(1-\nu)\nabla^2\chi - \chi_{,zz} \tag{7.1}$$

and

$$\begin{aligned} &\Sigma_r = (\nu\nabla^2\chi - \chi_{,rr})_{,z}, \qquad \Sigma_\theta = (\nu\nabla^2\chi - r^{-1}\chi_{,r})_{,z}, \\ &\Sigma_z = [(2-\nu)\nabla^2\chi - \chi_{,zz}]_{,z}, \qquad \Sigma = [(1-\nu)\nabla^2\chi - \chi_{,zz}]_{,r}. \end{aligned} \tag{7.2}$$

In these equations,

$$\Sigma_r + \Sigma_\theta + \Sigma_z = (1+\nu)\nabla^2\chi_{,z}, \qquad \nabla^2\chi = \chi_{,rr} + r^{-1}\chi_{,r} + \chi_{,zz} \tag{7.3}$$

and, if $(P/HR^2)e(z;\varepsilon)\delta(r)/2\pi r$ is the unknown upward vertical body force,

$$(1-\nu)\varepsilon\nabla^2\nabla^2\chi = -\varepsilon e(z;\varepsilon)\delta(r)/2\pi r. \tag{7.4}$$

(See Fung, 1965, p. 197.) Note that $\int_{1-\varepsilon}^{1+\varepsilon} e(z;\varepsilon)\mathrm{d}z = 1$.

The boundary conditions, as before, are that the inner and outer surfaces of the shell be stress free except for the point loads at $r = 0$, $z = \pm(1+\varepsilon)$. In circular cylindrical coordinates, these conditions imply that

$$\varepsilon\rho^{-1}S_\rho = r^2\Sigma + z^2\Sigma_z + 2rz\Sigma = \begin{cases} -\delta(r)/2\pi r, & z = \pm\sqrt{(1+\varepsilon)^2 - r^2} \\ 0, & z = \pm\sqrt{(1-\varepsilon)^2 - r^2} \end{cases}, \quad 0 \leqslant r \leqslant 1 \tag{7.5}$$

and

$$\varepsilon^{1/2}\rho^{-1}T = rz(\Sigma_r - \Sigma_z) + (z^2 - r^2)\Sigma = 0, \qquad z = \pm\sqrt{(1\pm\varepsilon)^2 - r^2}, \quad 0 \leqslant r \leqslant 1. \tag{7.6}$$

We henceforth focus on the upper half of the spherical shell.

Because all stresses vary as rapidly with radial distance as they do with meridional distance, we introduce the new scaled independent variables and stress function

$$r = \varepsilon s, \qquad z = 1 + \varepsilon t, \qquad \chi = \varepsilon^3\phi(s,t;\varepsilon), \quad -1 \leqslant t \leqslant 1. \tag{7.7}$$

Let

$$\Delta\phi \equiv \phi_{,ss} + s^{-1}\phi_{,s} + \phi_{,tt} \tag{7.8}$$

and

$$e(z;\varepsilon) = (1+\varepsilon t)^{-2}[(1/2) - F(t;\varepsilon)], \tag{7.9}$$

where $F - (1/2)$ is the dimensionless thickness distribution of the line load that produces the shell-like solution. Then, because $\varepsilon\delta(\varepsilon s) = \delta(s)$, Eqs. (7.1), (7.2), and (7.4) take the forms

$$2u_r = -\phi_{,st}, \qquad 2u_z = 2(1-\nu)\Delta\phi - \phi_{,tt}, \tag{7.10}$$

$$\begin{aligned} &\Sigma_r = (\nu\Delta\phi - \phi_{,ss})_{,t}, \qquad \Sigma_\theta = (\nu\Delta\phi - s^{-1}\phi_{,s})_{,t} \\ &\Sigma_z = [(2-\nu)\Delta\phi - \phi_{,tt}]_{,t}, \qquad \Sigma = [(1-\nu)\Delta\phi - \phi_{,tt}]_{,s} \equiv -\psi_{,s}, \end{aligned} \tag{7.11}$$

and

$$(1-\nu)\Delta\Delta\phi = (1+\varepsilon t)^{-2}[F(t;\varepsilon) - (1/2)]\delta(s)/2\pi s. \tag{7.12}$$

We now assume that all dependent variables have the asymptotic form

$$f(s,t;\varepsilon) = \overset{0}{f}(s,t) + \varepsilon\overset{1}{f}(s,t) + \mathrm{O}(\varepsilon^2). \tag{7.13}$$

Then, by Eq. (5.31), the basic differential equation (7.12) reads, to lowest order,

$$(1-\nu)\Delta\Delta\overset{0}{\phi} = 3(t^2-1)\delta(s)/8\pi s, \tag{7.14}$$

whereas the boundary conditions (7.5) and (7.6), to lowest order, read

$$[(2-\nu)\Delta\overset{0}{\phi} - \overset{0}{\phi}_{,tt}]_{,t} = \begin{cases} -\delta(s)/2\pi s, & t=1 \\ 0, & t=-1 \end{cases}, \quad 0 \leqslant s < \infty \tag{7.15}$$

and

$$\overset{0}{\psi} \equiv (1-\nu)\Delta\overset{0}{\phi} - \overset{0}{\phi}_{,tt} = 0, \quad 0 \leqslant s < \infty, \ \ t = \pm 1. \tag{7.16}$$

In this last equation, we have integrated with respect to s the expression for $\overset{0}{\Sigma}$ coming from the fourth term of Eq. (7.11) and discarded the resulting constant of integration. Eqs. (7.14)–(7.16) are equivalent to those for an infinite slab under a downward concentrated load at the origin and an upward, parabolically distributed, equilibrating line load. The solution may be represented in terms of inverse Hankel transforms, as in Tranter (1974).

Thus, let

$$\bar{\phi}(\lambda,t) = \mathscr{H}\{\overset{0}{\phi}(s,t)\} \equiv \int_0^\infty \overset{0}{\phi}(s,t)J_0(s\lambda)s\,\mathrm{d}s, \quad 0 \leqslant \lambda < \infty \tag{7.17}$$

denote the Hankel transform (of order zero) of $\overset{0}{\phi}$, where J_0 is the Bessel function of order zero of the first kind and λ is the transform variable. The (formal) inverse of Eq. (7.17) is

$$\overset{0}{\phi}(s,t) = \mathscr{H}^{-1}\{\bar{\phi}(\lambda,t)\} = \int_0^\infty \bar{\phi}(\lambda,t)J_0(\lambda s)\lambda\,\mathrm{d}\lambda, \quad 0 \leqslant s < \infty. \tag{7.18}$$

(Although λ is assumed, initially, to be positive, it may be necessary to move into the complex plane to obtain convergent integrals. However, we shall proceed formally.) For simplicity, we have dropped the index "0" on $\bar{\phi}$.

Noting that

$$\int_0^\infty \Delta\overset{0}{\phi}(s,t)J_0(s\lambda)s\,\mathrm{d}s = -\lambda^2\bar{\phi}(\lambda,t) + \bar{\phi}^{\bullet\bullet}(\lambda,t), \tag{7.19}$$

we have, on taking the Hankel transforms of Eqs. (7.14)–(7.16),

$$(1-\nu)(\lambda^4\bar{\phi} - 2\lambda^2\bar{\phi}^{\bullet\bullet} + \bar{\phi}^{\bullet\bullet\bullet\bullet}) = (3/8\pi)(t^2-1), \tag{7.20}$$

$$(2-\nu)\lambda^2\bar{\phi}^{\bullet}_{\pm} - (1-\nu)\bar{\phi}^{\bullet\bullet\bullet}_{\pm} = \begin{Bmatrix} 1/2\pi \\ 0 \end{Bmatrix}, \tag{7.21}$$

and

$$\bar{\psi} = (1-\nu)\lambda^2\bar{\phi}_{\pm} + \nu\bar{\phi}^{\bullet\bullet}_{\pm} = 0, \tag{7.22}$$

where a dot ($\bullet$) denotes differentiation with respect to t and $\bar{\phi}_{\pm} \equiv \bar{\phi}(\lambda,\pm 1)$.

The solution of Eq. (7.20) has the form

$$\bar{\phi} = A(\lambda;\nu)\,\mathrm{sh}\,\xi + B(\lambda;\nu)\,\mathrm{ch}\,\xi + C(\lambda;\nu)\xi\,\mathrm{sh}\,\xi + D(\lambda)\xi\,\mathrm{ch}\,\xi + \frac{3(4+\xi^2-\lambda^2)}{8(1-\nu)\pi\lambda^6}, \tag{7.23}$$

where $\xi = \lambda t$, sh $\equiv$ sinh, and ch $\equiv$ cosh. Application of the boundary conditions (7.21) and (7.22) yields

$$(2-\nu)[(A+D)\operatorname{ch}\lambda \pm (B+C)\operatorname{sh}\lambda \pm C\lambda\operatorname{ch}\lambda + D\lambda\operatorname{sh}\lambda] \pm \frac{3(2-\nu)}{4(1-\nu)\pi\lambda^5} - (1-\nu)[(A+3D)\operatorname{ch}\lambda \pm (B+3C)\operatorname{sh}\lambda \pm C\lambda\operatorname{ch}\lambda + D\lambda\operatorname{sh}\lambda] = \begin{Bmatrix} 1/2\pi\lambda^3 \\ 0 \end{Bmatrix} \tag{7.24}$$

and

$$(1-\nu)(\pm A\operatorname{sh}\lambda + B\operatorname{ch}\lambda + C\lambda\operatorname{sh}\lambda \pm D\lambda\operatorname{ch}\lambda) + \frac{3}{2\pi\lambda^6} + \nu[\pm(A+2D)\operatorname{sh}\lambda + (B+2C)\operatorname{ch}\lambda + C\lambda\operatorname{sh}\lambda \pm D\lambda\operatorname{ch}\lambda] + \frac{3\nu}{4(1-\nu)\pi\lambda^6} = 0. \tag{7.25}$$

Adding and subtracting first Eq. (7.24) at $\xi = \pm\lambda$ and then (7.25) at $\xi = \pm\lambda$, we obtain the two sets of simultaneous equations

$$\begin{aligned} A\operatorname{ch}\lambda + D[\lambda\operatorname{sh}\lambda - (1-2\nu)\operatorname{ch}\lambda] &= \frac{1}{4\pi\lambda^3}, \\ A\operatorname{sh}\lambda + D(\lambda\operatorname{ch}\lambda + 2\nu\operatorname{sh}\lambda) &= 0 \end{aligned} \tag{7.26}$$

and

$$B\operatorname{sh}\lambda + C[\lambda\operatorname{ch}\lambda - (1-2\nu)\operatorname{sh}\lambda] = \frac{(1-\nu)\lambda^2 - 3(2-\nu)}{4(1-\nu)\pi\lambda^5}, \tag{7.27}$$

$$B\operatorname{ch}\lambda + C(\lambda\operatorname{sh}\lambda + 2\nu\operatorname{ch}\lambda) = -\frac{3(2-\nu)}{4(1-\nu)\pi\lambda^6}. \tag{7.28}$$

Thus,

$$4\pi\lambda^3(\lambda + \operatorname{sh}\lambda\operatorname{ch}\lambda)\{A, D\} = \{\lambda\operatorname{ch}\lambda + 2\nu\operatorname{sh}\lambda, -\operatorname{sh}\lambda\} \tag{7.29}$$

and

$$4\pi\lambda^6(\lambda - \operatorname{sh}\lambda\operatorname{ch}\lambda)\{B, C\} = 3(2-\nu)(1-\nu)^{-1}\{\lambda^2\operatorname{sh}\lambda + (1-2\nu)(\operatorname{sh}\lambda - \lambda\operatorname{ch}\lambda), \operatorname{sh}\lambda - \lambda\operatorname{ch}\lambda\} + \lambda^3\{-(\lambda\operatorname{sh}\lambda + 2\nu\operatorname{ch}\lambda), \operatorname{ch}\lambda\}. \tag{7.30}$$

Note that A and D are *odd* in λ whereas B and C are *even*, so that, setting $\xi = \lambda t$, we see from Eq. (7.23) that $\bar{\phi}$ is *even* in λ. This simple observation has an important (but expected) consequence: $\overset{0}{\phi}(s,t;\nu)$ is transcendentally small as $s \to \infty$. This follows immediately from the formal asymptotic expansion (Tranter, 1974, p. 67)

$$\int_0^\infty f(\lambda)J_0(s\lambda)\mathrm{d}\lambda = \frac{f(0)}{s} - \frac{1}{2}\frac{f''(0)}{s^3} + \mathrm{O}\left(\frac{1}{s^5}\right), \tag{7.31}$$

where, in our case, $f = \lambda\bar{\phi}(\lambda, t; \nu)$ and is odd in λ, so that every derivative on the right side of Eq. (7.31) vanishes.

7.1. Solution near the polar axis

There is one final item we need to consider: the behavior of the lowest-order slab-like solutions near the concentrated loads on the polar axis of the shell. This is governed by the large λ behavior of the Hankel transform $\bar{\phi}(\lambda, t; \nu)$.

Thus, from Eqs. (7.29) and (7.30),

$$\{A, D\} \sim \{B, C\} \sim \frac{\mathrm{e}^{-\lambda}}{2\pi\lambda^3}\{\lambda + 2\nu, -1\} \quad \text{as } \lambda \to \infty, \tag{7.32}$$

so that Eq. (7.23) has the asymptotic form

$$\bar{\phi} \sim \frac{(1-t)\mathrm{e}^{-\lambda(1-t)}}{2\pi\lambda^2} + \frac{3[4 + \lambda^2(t^2 - 1)]}{8(1-\nu)\pi\lambda^6} \quad \text{as } \lambda \to \infty. \tag{7.33}$$

Because our main interest is the behavior of the stresses near the concentrated loads, we have, from the first term of Eq. (7.3), Eqs. (7.7), (7.19), (7.22), and (7.33), that as $\lambda \to \infty$,

$$\bar{\Sigma}_r^0 + \bar{\Sigma}_\theta^0 + \bar{\Sigma}_z^0 = (1+\nu)(\bar{\phi}^{\bullet\bullet\bullet} - \lambda^2\bar{\phi}^{\bullet}) \sim -\frac{1+\nu}{\pi}\left[\mathrm{e}^{-\lambda(1-t)} + \frac{3t}{4(1-\nu)\lambda^2}\right] \tag{7.34}$$

and

$$\bar{\psi} \sim (1/2\pi)(1-t)\mathrm{e}^{-\lambda(1-t)} + \frac{3[4 - 2\nu + (1-\nu)\lambda^2(t^2-1)]}{8\pi(1-\nu)\lambda^4}. \tag{7.35}$$

Strictly speaking, the inverse transforms of λ^{-2} and λ^{-4} do not exist. However, because

$$\mathscr{H}\{\delta(s)/s\} = 1 \quad \text{and} \quad \mathscr{H}\{\mathscr{L}f(s)\} = -\lambda^2\mathscr{H}\{f(s)\}, \tag{7.36}$$

where $\mathscr{L}f = s^{-1}[sf'(s)]'$ as in Eq. (5.5), we can identify the inverse transforms of λ^{-2} and λ^{-4} with functions $f(s)$ and $g(s)$ such that

$$\mathscr{L}f = -\delta(s)/s \quad \text{and} \quad \mathscr{L}^2 g = \delta(s)/s. \tag{7.37}$$

That is,

$$f = -\ln s + c_1 \tag{7.38}$$

and

$$g = (1/4)s^2\ln s + c_2 s^2 + c_3\ln s + c_4, \tag{7.39}$$

where $c_1, \ldots, c_4$ are unknown constants. As c_1 and c_2 have no influence on the dominant behavior of f and g as $s \to 0$, we set them to zero. Moreover, because $\Sigma^0 = \overset{0}{\psi}_{,s}$, we may ignore c_4.

From Eq. (4) on p. 29 of Erdélyi et al. (1954),

$$\Sigma_r^0 + \Sigma_\theta^0 + \Sigma_z^0 \sim -\frac{1+\nu}{\pi}\left\{\frac{1-t}{[s^2 + (1-t)^2]^{3/2}} - \frac{3t\ln s}{4(1-\nu)}\right\} \quad \text{as } s \to 0 \tag{7.40}$$

and

$$\overset{0}{\psi} \sim \frac{(1-t)^2}{2\pi[s^2 + (1-t)^2]^{3/2}} + \frac{1}{8(1-\nu)\pi}[2c_3 + (1-\nu)(1-t^2)]\ln s \quad \text{as } s \to 0. \tag{7.41}$$

To determine c_3, note from overall vertical equilibrium that

$$\int_{-1}^{1}\Sigma^0(s,t)\,\mathrm{d}t = 0 \quad \text{or} \quad \int_{-1}^{1}\overset{0}{\psi}(s,t)\,\mathrm{d}t = 0, \tag{7.42}$$

because the concentrated surface and body forces in the slab-like solution are self-equilibrating. As the second term of Eq. (7.42) must hold, in particular, in the limit as $s \to 0$, we find, on using Eq. (7.41), that $c_3 = (2/3)(1 - v)$.

The first terms on the right of Eqs. (7.40) and (7.41) agree with the Boussinesq solution for a semi-infinite elastic solid under a concentrated normal load on its boundary (Timoshenko and Goodier, 1970, p. 401).

7.2. Stress resultants and couples

We end our analysis by considering the sum of the principle stress resultants and couples, $(P/H)(\overset{0}{N}, \overset{0}{M})$, associated with the lowest-order slab-like solutions, namely,

$$(\overset{0}{N}, \overset{0}{M}) = \int_{-1}^{1} (1, t)[\Sigma_r^0(s,t) + \Sigma_\theta^0(s,t)]\,\mathrm{d}t. \tag{7.43}$$

Taking Hankel transforms (and dropping the index "0"), using the first and second terms of Eq. (7.11), and integrating, we obtain

$$\bar{N} = (1 - 2v)\lambda^2(\bar{\phi}_+ - \bar{\phi}_-) + 2v(\bar{\phi}_+^{\bullet\bullet} - \bar{\phi}_-^{\bullet\bullet}) \tag{7.44}$$

and

$$\bar{M} = (1 - 2v)\lambda^2(\bar{\phi}_+ + \bar{\phi}_- - \bar{\Phi}) + 2v(\bar{\phi}_+^{\bullet\bullet} + \bar{\phi}_-^{\bullet\bullet} + \bar{\phi}_-^{\bullet} - \bar{\phi}_+^{\bullet}), \tag{7.45}$$

where $\bar{\Phi} \equiv \int_{-1}^{1} \bar{\phi}\,\mathrm{d}t$.

To simplify these expressions, we first use the boundary conditions (7.22) to eliminate $\bar{\phi}_\pm^{\bullet\bullet}$ so that Eqs. (7.44) and (7.45) reduce to

$$\bar{N} = -\lambda^2(\bar{\phi}_+ - \bar{\phi}_-) \tag{7.46}$$

and

$$\bar{M} = -[\lambda^2(\bar{\phi}_+ + \bar{\phi}_-) + 2v(\bar{\phi}_+^{\bullet} - \bar{\phi}_-^{\bullet}) + (1 - 2v)\lambda^2\bar{\Phi}]. \tag{7.47}$$

To eliminate $\bar{\Phi}$, note that the fourth term of Eq. (7.11) and the second term of Eq. (7.42) imply that

$$\lambda^2\bar{\Phi} = v(1 - v)^{-1}(\bar{\phi}_-^{\bullet} - \bar{\phi}_+^{\bullet}). \tag{7.48}$$

Hence,

$$\bar{M} = -[\lambda^2(\bar{\phi}_+ + \bar{\phi}_-) + v(1 - v)^{-1}(\bar{\phi}_+^{\bullet} - \bar{\phi}_-^{\bullet})]. \tag{7.49}$$

Substituting Eq. (7.23) into Eqs. (7.46) and (7.49) and use of Eq. (7.32) yields

$$\bar{N} = -2\lambda^2(A\,\mathrm{sh}\,\lambda + D\lambda\,\mathrm{ch}\,\lambda) = -\frac{v\,\mathrm{sh}^2\lambda}{\pi\lambda(\lambda + \mathrm{sh}\,\lambda\,\mathrm{ch}\,\lambda)} \sim -\frac{v}{\pi\lambda} \quad \text{as } \lambda \to \infty \tag{7.50}$$

and

$$\begin{aligned}\bar{M} = &-2(1 - v)^{-1}\lambda\{B[(1 - v)\lambda\,\mathrm{ch}\,\lambda + v\,\mathrm{sh}\,\lambda] + C[(1 - v)\lambda^2\,\mathrm{sh}\,\lambda + v(\mathrm{sh}\,\lambda + \lambda\,\mathrm{ch}\,\lambda)]\} \\ &- (3/2\pi)(2 - v)(1 - v)^{-2}\lambda^{-4} \sim -\frac{v}{\pi\lambda} - \frac{3(2 - v)}{2(1 - v)^2\pi\lambda^4} \quad \text{as } \lambda \to \infty.\end{aligned} \tag{7.51}$$

Because $\mathscr{H}^{-1}\{\lambda^{-1}\} = s^{-1}$ and $\mathscr{H}^{-1}\{\lambda^{-4}\}$ may be identified with a function of the form (7.39),

$$\overset{0}{N} \sim -v/\pi s \quad \text{and} \quad \overset{0}{M} \sim -v/\pi s \ \text{ as } s \to 0. \tag{7.52}$$

The appearance of a Poisson ratio factor in these expressions strongly suggests a "normal stress" effect. This is nor surprising considering that it is the normal stresses that must ultimately dominate as we approach the points of application of the concentrated surface loads.

8. Conclusions

We have shown that three different scalings of the dependent and independent variables of the governing linear equations for an elastically isotropic spherical shell under equal and opposite concentrated surface loads suggest that the exact solution is the sum of (i) a membrane-like solution, (ii) a (shallow) shell-like solution, and (iii) a slab-like solution. Gregory et al. (1999) have presented a simple, exact formulas for (i) and exact eigenfunction representations for (ii). They also have expanded the latter for small values of H/R (the half thickness to mid-surface radius ratio) to obtain explicit first-order corrections to solutions of the classical Kirchhoff–Love shell equations. They left (iii) untouched due to the complicated form of the solutions. In the present paper, we obtained asymptotic solutions for (ii) and (iii) – (i) was simple enough that no asymptotics were needed – by working directly with the governing differential equations rather than with their solutions. This not only made the calculations in (ii) less laborious, but it allowed us to obtain explicit integral representations (inverse Hankel transforms) for the lowest-order solutions in (iii). These, in turn, allowed us to determine the correct behavior of the sum of the principal stress resultants and couples in the immediate neighborhood of the concentrated loads.

Our analysis leaves open several questions, the most important of which, in our opinion, is: can one show *rigorously* that the exact solution of the original problem is the unique sum of (i), (ii), and (iii)? We note that Gregory (1992) has proved such a result for the stretching of elastically isotropic plates (where, of course, there is neither a membrane-like component nor a shell-like component). Rather than attempting to extend Gregory's approach, which makes use of the specific form of various types of three-dimensional solutions, we conjecture that it might be possible to find a (more or less) direct way to characterize the dimensionless self-equilibrating line load $F(\zeta;\varepsilon)$ introduced following equation (4.29) that appears in the equations for the shell-like solutions. All we have shown herein in Eq. (5.31) is that $F(\zeta;0)$ is parabolic.

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